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BUCKLING UNDER EXTERNAL PRESSURE OF CYLINDRICAL SHELLS WITH DIMPLE SHAPED INITIAL IMPERFECTIONS†

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Abstract—The approximate load–deflection behavior of a simply supported cylindrical shell subjected to external pressure is determined for the case where the shell has a dimple shaped initial deflection. This is accomplished by the use of a "two-timing" perturbation expansion applied to the Kármán–Donnell shell equations. We show that if the shell is imperfection-sensitive to an initial deflection in the shape of the linear buckling mode, then it is also imperfection-sensitive to a dimple imperfection. The degradation in buckling load for the dimple imperfection depends linearly upon the initial deflection amplitude ε whereas for the modal imperfection it is proportional to the two-thirds power of ε.

1. INTRODUCTION

A NUMBER of important shell structures have their "classical" buckling loads greatly reduced by the presence of small geometrical imperfections and such shells are said to be imperfection-sensitive. Koiter's general theory of elastic stability [1, 2] provides a basis for determining the imperfection-sensitivity of a structure with respect to initial displacements having the same shape as their "classical" buckling mode and a number of shells have been treated in this way. References can be found in the survey paper by Budiansky and Hutchinson [3]. The question arises as to whether structures that are imperfection-sensitive to modal initial deflection are also imperfection-sensitive to non-modal initial deflections. In a recent investigation, Amazigo *et al.* [4] have shown that the beam on a nonlinear softening elastic foundation which is imperfection-sensitive to modal initial deflection, is also imperfection-sensitive to dimple shaped initial deflections. They solve this problem in two ways; first using the method of equivalent linearization and second a "two-timing" perturbation expansion technique. Both methods give the same asymptotic result for buckling load versus initial deflection amplitude for small values of the later quantity.

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In this paper we will use a two-timing perturbation expansion to obtain the load–deflection behavior of a simply supported circular cylinder with a dimple shaped initial deflection subjected to external pressure. Budiansky and Amazigo [5] have shown, using Koiter's theory, that these shells are imperfection-sensitive to modal imperfections over a large range of a geometrical parameter $Z = (L^2/Rh)\sqrt{(1-\nu^2)}$. The present analysis shows that these shells are imperfection-sensitive to dimple shaped imperfections over the same range of the parameter Z, but whereas for modal imperfection the loss in buckling strength is of the order of the two-thirds power of the initial deflection amplitude ε in this case the loss in buckling strength depends linearly upon ε. A similar situation was found in the case of the beam on the cubic foundation [4]. In the Appendix it is shown that an alternative perturbation expansion can also be used to obtain the results in [5].

2. KÁRMÁN–DONNELL EQUATIONS FOR CIRCULAR CYLINDRICAL SHELLS

Since the Kármán–Donnell equations are fairly well known, we will set down immediately the nondimensionalized form of these equations that form the starting point for the present analysis. Notice that the linear prebuckling solution of the Kármán–Donnell equations has already been removed from the nondimensional displacement and stress function, w and f, that appear in these equations which are

$$\bar{\nabla}^4 f-(1+\zeta)^2 w_{,xx} = -(1+\zeta)^2 H\{\tfrac{1}{2}S(w,w)+S(w_0,w)\} \tag{1}$$

$$\bar{\nabla}^4 w-K(\zeta)f_{,xx}+\lambda(\tfrac{1}{2}\alpha w_{,xx}+\zeta w_{,yy}) = -K(\zeta)H\{S(w,f)+S(w_0,f)\}-\lambda(\tfrac{1}{2}\alpha w_{0,xx}+\zeta w_{0,yy}) \tag{2}$$

where

$$\bar{\nabla}^4 \equiv \left(\frac{\partial^2}{\partial x^2}+\zeta\frac{\partial^2}{\partial y^2}\right)^2, \qquad S(P,Q) = P_{,xx}Q_{,yy}-2P_{,xy}Q_{,xy}+P_{,yy}Q_{,xx},$$

and

$$K(\zeta) = [(1+\zeta)^2-\lambda_c(\zeta+\tfrac{1}{2}\alpha)]. \tag{3}$$

The nondimensionalization has been carried out with respect to the form of the classical buckling solution as given by Batdorf [6]. We will now relate the above nondimensional quantities to the dimensional quantities for a thin shell of length L, radius R and thickness h, made of elastic material with Young's modulus E, Poisson's ratio ν and bending stiffness $D = Eh^3/12(1-\nu^2)$, and subjected to an external normal pressure p. If X, Y are cartesian coordinates in the axial and circumferential direction then they are related to the nondimensional coordinates x, y by

$$Y = \frac{Ry}{n}, \qquad X = \frac{Lx}{\pi} \tag{4}$$

where the integer n is the number of circumferential buckle waves.

The actual stress resultants are related to an Airy stress function F by

$$N_x = F_{,YY}, \qquad N_y = F_{,XX}, \qquad N_{xy} = -F_{,XY} \tag{5}$$

and F and the radial displacement W and initial stress free radial displacement $\overline{W}$ are related to f, w and w_0 by

$$F = -\tfrac{1}{2}pR(X^2+\tfrac{1}{2}\alpha Y^2)+\frac{Eh^2L^2}{\pi^2R(1+\zeta)^2}f \tag{6}$$

$$= -\tfrac{1}{2}pR(X^2+\tfrac{1}{2}\alpha Y^2)-\frac{\pi^2DRh}{L^2}K(\zeta)f, \tag{7}$$

and

$$W = (pR^2/Eh)(1-\nu\tfrac{1}{2}\alpha)+hw, \tag{8}$$

and

$$\overline{W} = hw_0. \tag{9}$$

The first terms in (6)–(8) are due to the linear prebuckling solution of the Kármán–Donnell equations and represent a uniform stress field producing a constant radial displacement. Quantity α takes the value $\alpha = 1$ if the pressure contributes to axial stress through end plates, and $\alpha = 0$ if pressure only acts laterally. Parameters λ, ζ and H are defined by

$$\lambda = \frac{pL^2R}{\pi^2D}, \tag{10}$$

$$\zeta = \left(\frac{nL}{\pi R}\right)^2, \tag{11}$$

$$H = \frac{n^2h}{R}. \tag{12}$$

When the load parameter $\lambda = \lambda_c$ the pressure $p = p_c$, the classical buckling pressure as given by Batdorf [6]. Thus

$$\lambda_c = (\zeta+\tfrac{1}{2}\alpha)^{-1}[(1+\zeta)^2+A^2(1+\zeta)^{-2}], \tag{13}$$

where

$$A = \frac{L^2}{\pi^2hR}\sqrt{[12(1-\nu^2)]}, \tag{14}$$

and n in expression (11) for ζ is the integer that minimizes λ_c. If λ_c in (13) is minimized on the basis of the simplifying assumption that ζ varies continuously (cf. Ref. [6] for a discussion of this assumption) then

$$\lambda_c = \frac{4(1+\zeta)^2}{3\zeta+1+\alpha}, \qquad A^2 = \frac{(1+\zeta)^4(\zeta-1+\alpha)}{3\zeta+1+\alpha}. \tag{15}$$

With λ_c defined by equation (13), the two expressions (6) and (7) for F are of course identical. It is however convenient to display both expressions since (6) is used in the nondimensionalization leading to (1) and (7) is used in obtaining (2).

We will also require expressions for bending strains $\kappa_x, \kappa_y, \kappa_{xy}$ and membrane strains $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$, which in terms of dimensional displacements U (axial), V (circumferential) and W are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} U_{,X} \\ V_{,Y}+W/R \\ \frac{1}{2}(U_{,Y}+V_{,X}) \\ -W_{,XX} \\ -W_{,YY} \\ -W_{,XY} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} (W_{,X})^2 \\ (W_{,Y})^2 \\ W_{,X}W_{,Y} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

and the stress–strain relations

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} C & \nu C & & & & \\ \nu C & C & & & & \\ & & (1-\nu)C & & & \\ & & & D & \nu D & \\ & & & \nu D & D & \\ & & & & & (1-\nu)D \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{17}$$

where M_x, M_y, M_{xy} are the dimensional stress couples and $C = Eh/(1-\nu)^2$.

The usual simply supported boundary conditions will be assumed at the ends of the shell, namely zero normal moment $M_x = 0$, zero circumferential displacement $V = 0$, radial displacement $W = (pR^2/Eh)(1-\nu\frac{1}{2}\alpha)$ and membrane stress $N_x = -\frac{1}{2}\alpha pR$, at $X = 0, L$. This leads us, using (5), (16), (17) and the normalization (4), (7) and (8), to the boundary conditions on w, f which are

$$f = f_{,xx} = w = w_{,xx} = 0 \quad \text{at } x = 0, \pi. \tag{18}$$

3. "TWO-TIMING" PERTURBATION EXPANSION

The application of the two-timing technique (see, for example [7, 8] for a discussion of this technique) to this problem follows closely its application to the corresponding problem for the column in Ref. [4]. We consider the shell to have an initial stress free radial displacement in the nondimensional form

$$w_0(x, y) = \varepsilon W_0(y) \sin x \tag{19}$$

where ε is a small parameter characterizing the amplitude of the initial deflection and $W_0(y)$ is defined on the infinite interval $-\infty < y < \infty$ to be continuously differentiable and to satisfy the exponential decay condition

$$|W_0(y)| < M\,e^{-\gamma|y|} \qquad (M, \gamma > 0). \tag{20}$$

In other words, in order to obtain a solution, we shall have to disregard the fact that the cylinder is closed and that all functions must be periodic in the circumferential coordinate y.

This is equivalent to the previously made assumption that ζ be a continuous variable. However we require that the displacement w and the stress function f be bounded for $|y| \to \infty$.

By analogy with the column problem [4] we anticipate that the buckling deformation will behave in the y coordinate as a periodic function that is slowly decreasing in amplitude as $|y| \to \infty$. This suggests introducing a further length coordinate as independent variable, this new stretched coordinate being associated with the amplitude decay rate. Following [4], we introduce the variable $z \equiv \eta y$, where η is a load parameter defined by

$$\eta^2 = 1 - \lambda/\lambda_c \tag{21}$$

and w and f are now considered to be functions of the three independent variables x, y and z so that

$$\frac{\partial}{\partial y} \text{ is replaced by } \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial z} \tag{22}$$

and

$$\bar{\nabla}^4 \text{ is replaced by } \left(\frac{\partial^2}{\partial x^2} + \zeta \frac{\partial^2}{\partial y^2} + 2\eta\zeta \frac{\partial^2}{\partial y \partial z} + \eta^2 \zeta \frac{\partial^2}{\partial z^2}\right)^2 = \left(\bar{\nabla}^2 + 2\eta\zeta \frac{\partial^2}{\partial y \partial z} + \eta^2 \zeta \frac{\partial^2}{\partial z^2}\right)^2. \tag{23}$$

We use η as an expansion parameter, and write

$$\begin{Bmatrix} w \\ f \end{Bmatrix} = \sum_{m=1}^{\infty} \eta^m \begin{Bmatrix} w_m(x, y, z) \\ f_m(x, y, z) \end{Bmatrix}, \tag{24}$$

$$(\lambda/\lambda_c)\varepsilon = B_1 \eta + B_2 \eta^2 + B_3 \eta^3 + \ldots. \tag{25}$$

Note, that if

$$\begin{Bmatrix} P \\ Q \end{Bmatrix} = \eta \begin{Bmatrix} P_1 \\ Q_1 \end{Bmatrix} + \eta^2 \begin{Bmatrix} P_2 \\ Q_2 \end{Bmatrix} + \ldots, \tag{26}$$

then

$$\begin{aligned} S(P, Q) = {} & \eta^2 S(P_1, Q_1) + \eta^3 \{S(P_1, Q_2) + S(P_2, Q_1) \\ & + 2(P_{1,yz} Q_{1,xx} - P_{1,xz} Q_{1,xy} - Q_{1,xz} P_{1,xy} + Q_{1,yz} P_{1,xx})\} + 0(\eta^4). \end{aligned} \tag{27}$$

On substituting from equation (19) for w_0, and using relations (21)–(27) in equations (1) and (2), and then equating to zero the coefficients of successive powers of η, we find that the first three perturbation equations are

$$\begin{aligned} L^{(1)}(f_1, w_1) &= 0, \\ L^{(2)}(f_1, w_1) &= \lambda_c[\tfrac{1}{2}\alpha W_0(y) - \zeta W_0''(y)] B_1 \sin x, \end{aligned} \tag{28}$$

$$\begin{aligned} L^{(1)}(f_2, w_2) = {} & -4\zeta \frac{\partial^2}{\partial y \partial z}(\bar{\nabla}^2 f_1) - (1+\zeta)^2 H\{\tfrac{1}{2} S(w_1, w_1) - B_1 S_0(w_1)\}, \\ L^{(2)}(f_2, w_2) = {} & \lambda_c[\tfrac{1}{2}\alpha W_0(y) - \zeta W_0''(y)] B_2 \sin x \\ & - 4\zeta \frac{\partial^2}{\partial y \partial z}(\bar{\nabla}^2 + \tfrac{1}{2}\lambda_c) w_1 - HK(\zeta)\{S(w_1, f_1) - B_1 S_0(f_1)\}, \end{aligned} \tag{29}$$

and

$$
\begin{aligned}
L^{(1)}(f_3, w_3) = & -4\zeta\frac{\partial^2}{\partial y\partial z}(\bar{\nabla}^2 f_2) - 2\zeta\frac{\partial^2}{\partial z^2}\left(\bar{\nabla}^2 + 2\zeta\frac{\partial^2}{\partial y^2}\right) f_1 - (1+\zeta)^2 H\{S(w_1, w_2) \\
& + 2(w_{1,yz}w_{1,xx} - w_{1,xz}w_{1,xy}) - B_1[S_0(w_2) + 2W_0(y)w_{1,yz}\sin x \\
& + 2W_0'(y)w_{1,xz}\cos x] - B_2 S_0(w_1)\}, \\
L^{(2)}(f_3, w_3) = & \lambda_c[\tfrac{1}{2}\alpha W_0(y) - \zeta W_0''(y)]B_3 \sin x + \lambda_c(\tfrac{1}{2}\alpha W_{1,xx} + \zeta w_{1,yy}) \\
& -4\zeta\frac{\partial^2}{\partial y\partial z}(\bar{\nabla}^2 + \tfrac{1}{2}\lambda_c)w_2 - 2\zeta\frac{\partial^2}{\partial z^2}\left(\bar{\nabla}^2 + 2\zeta\frac{\partial^2}{\partial y^2} + \tfrac{1}{2}\lambda_c\right) w_1 \\
& - HK(\zeta)\{S(w_1, f_2) + S(f_1, w_2) + 2(w_{1,yz}f_{1,xx} - w_{1,xz}f_{1,xy} \\
& + f_{1,yz}w_{1,xx} - f_{1,xz}w_{1,xy}) - B_1(S_0(f_2) + 2W_0(y)f_{1,yz}\sin x \\
& + 2W_0'(y)f_{1,xz}\cos x) - B_2 S_0(f_1)\},
\end{aligned}
\tag{30}
$$

where $L^{(1)}, L^{(2)}$ are linear operators defined by

$$
\begin{aligned}
L^{(1)}(f_m, w_m) &= \bar{\nabla}^4 f_m - (1+\zeta)^2 w_{m,xx}, \\
L^{(2)}(f_m, w_m) &= \bar{\nabla}^4 w_m - K(\zeta) f_{m,xx} + \lambda_c(\tfrac{1}{2}\alpha w_{m,xx} + \zeta w_{m,yy}),
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
S_0(P) &= -(1/\varepsilon)S(w_0, P) \\
&= W_0(y)P_{,yy}\sin x + 2W_0'(y)P_{,xy}\cos x - W_0''(y)P_{,xx}\sin x
\end{aligned}
\tag{32}
$$

and of course $(\ )' \equiv \dfrac{\mathrm{d}}{\mathrm{d}y}(\ )$.

The boundary conditions (18) become

$$f_m = f_{m,xx} = w_m = w_{m,xx} = 0 \quad \text{at } x = 0, \pi$$

for terms corresponding to $m = 1, 2, 3, \ldots$ in the expansion (24). Guided by the analyses in [4], we admit the possibility of jumps in w_m, f_m or their derivatives at $y = 0$ and $z = 0$; however the physical quantities such as deflection, slope, shear, membrane stresses and moments must be continuous in y. (They must also, of course, be continuous in $0 < x < \pi$ but it is not necessary to invoke this condition explicitly.) Thus w and f and their first three circumferential derivatives must be continuous so that using (22) and expansion (24) we have that the following combinations of functions must be continuous in y and z:

$$
\left\{
\begin{array}{l}
u_m \\
\dfrac{\partial u_m}{\partial y} + \dfrac{\partial u_{m-1}}{\partial z} \\
\dfrac{\partial^2 u_m}{\partial y^2} + 2\dfrac{\partial^2 u_{m-1}}{\partial y\partial z} + \dfrac{\partial^2 u_{m-2}}{\partial z^2} \\
\dfrac{\partial^3 u_m}{\partial y^3} + 3\dfrac{\partial^3 u_{m-1}}{\partial y^2\partial z} + 3\dfrac{\partial^3 u_{m-2}}{\partial y\partial z^2} + \dfrac{\partial^3 u_{m-3}}{\partial z^3}
\end{array}
\right\} (m = 1, 2, 3, \ldots)
\tag{33}
$$

where

$$u_k = 0 \text{ for } k \le 0 \text{ and } u_k = \begin{Bmatrix} f_k \\ w_k \end{Bmatrix}, k = 1, 2, 3, \ldots .$$

We are now ready to solve the perturbation equations (28)–(30) successively in the usual manner. The right hand sides of successive equations are determined by the solutions of the previous equations, and the coefficients $B_1, B_2, B_3, \ldots$ as well as the differential equations governing the z dependence of the solutions are determined by eliminating secular terms (i.e. particular solutions arising from nonhomogeneous terms proportional to the solution of the homogeneous equations). These secular terms lead to unbounded solution.

The method of eliminating the secular terms is as follows: we note that the homogeneous solution to the operators on the left in equations (28)–(30), that we will be concerned with in this problem is

$$\begin{aligned} f &= a\,e^{\pm iy} \sin x \\ w &= b\,e^{\pm iy} \sin x \end{aligned} \tag{34}$$

with $a = -b$ and we are concerned with eliminating terms proportional to $e^{\pm iy} \sin x$ from the right hand sides of the equations. Let the coefficient of these terms be Ω_1 and Ω_2 and write

$$\begin{aligned} L^{(1)}(f, w) &= \Omega_1\, e^{\pm iy} \sin x \\ L^{(2)}(f, w) &= \Omega_2\, e^{\pm iy} \sin x \end{aligned} \tag{35}$$

Now, seeking a solution to (35) in the form (34) we find, using (31), that

$$[(1+\zeta)^2 a + (1+\zeta)^2 b]\, e^{\pm iy} \sin x = \Omega_1\, e^{\pm iy} \sin x, \tag{36}$$

$$[K(\zeta)a + K(\zeta)b]\, e^{\pm iy} \sin x = \Omega_2\, e^{\pm iy} \sin x, \tag{37}$$

and dividing (36) by (37) gives

$$\frac{(1+\zeta)^2}{K(\zeta)} = \frac{\Omega_1}{\Omega_2},$$

or

$$(1+\zeta)^2 \Omega_2 - K(\zeta)\Omega_1 = 0 \tag{38}$$

as the condition for suppressing secular terms of this type.

4. SOLUTION OF FIRST ORDER PERTURBATION EQUATION

The real-valued solution of (28) is

$$\begin{aligned} f_1 &= [a_1(z)\, e^{iy} + \bar{a}_1(z)\, e^{-iy}] \sin x + g_1(y) \sin x \\ w_1 &= -[a_1(z)\, e^{iy} + \bar{a}_1(z)\, e^{-iy}] \sin x + h_1(y) \sin x \end{aligned} \tag{39}$$

where $(\bar{\ })$ means the complex conjugate of ( ), the term involving $a(z), \bar{a}(z)$ is the bounded solution of the homogeneous part of (28), and the last term on the right of (39) is a particular solution of (28). Thus $g_1(y), h_1(y)$ (functions of y but not of z) is a particular solution of

$$\begin{aligned}&\left(\zeta\frac{d^2}{dy^2}-1\right)^2 g_1(y)+(1+\zeta)^2h_1(y) = 0\\&\left(\zeta\frac{d^2}{dy^2}-1\right)^2 h_1(y)+K(\zeta)g_1(y)+\lambda_c[\zeta h_1''(y)-\tfrac{1}{2}\alpha h_1(y)] = -\lambda_c[\zeta W_0''(y)-\tfrac{1}{2}\alpha W_0(y)]B_1.\end{aligned} \tag{40}$$

We now insist that $g_1(y), h_1(y)$ have bounded Fourier transforms

$$\begin{Bmatrix}\tilde{g}_1(\omega)\\ \tilde{h}_1(\omega)\end{Bmatrix} = \int_{-\infty}^{\infty}\begin{Bmatrix}g_1(y)\\ h_1(y)\end{Bmatrix} e^{i\omega y}\,dy \tag{41}$$

but admit the possibility of jumps

$$\begin{aligned}[g_1(0)] &= g_1(0^+)-g_1(0^-), \text{ etc.}\\ [h_1(0)] &= h_1(0^+)-h_1(0^-), \text{ etc.}\end{aligned} \tag{42}$$

Then the transform of (40) is

$$\begin{bmatrix}(\omega^2\zeta+1)^2 & (1+\zeta)^2\\ [(\zeta+1)^2-\lambda_c(\zeta+\frac{1}{2}\alpha)] & [(\omega^2\zeta+1)^2-\lambda_c(\omega^2\zeta+\frac{1}{2}\alpha)]\end{bmatrix}\begin{bmatrix}\tilde{g}_1(\omega)\\ \tilde{h}_1(\omega)\end{bmatrix} = \begin{bmatrix}\alpha_1(\omega)\\ \beta_1(\omega)\end{bmatrix}, \tag{43}$$

where

$$\begin{aligned}\alpha_1(\omega) &= \zeta^2[g_1'''(0)]-\zeta^2 i\omega[g_1''(0)]-\zeta(\omega^2\zeta+2)([g_1'(0)]-i\omega[g_1(0)]),\\ \beta_1(\omega) &= \zeta^2[h_1'''(0)]-\zeta^2 i\omega[h_1''(0)]-\zeta(\omega^2\zeta+2)([h_1'(0)]-i\omega[h_1(0)])\\ &\quad+\lambda_c\zeta([h_1'(0)]-i\omega[h_1(0)])+\lambda_c(\omega^2\zeta+\tfrac{1}{2}\alpha)B_1\tilde{W}_0(\omega).\end{aligned} \tag{44}$$

When λ_c takes on its minimized value (15), the determinant Δ of the coefficient matrix in (43) can be factored as follows:

$$\Delta = \zeta^2(\omega^2-1)^2\begin{vmatrix}(\omega^2-1)\zeta+\frac{3}{2}(1+\zeta) & -[(\omega^2-1)\zeta+2(1+\zeta)]\\ (\omega^2-1)\zeta+2(1+\zeta) & -\lambda_c\end{vmatrix} \tag{45}$$

[In fact when λ_c is given by (13) Δ has single zeros at $\omega = \pm 1$, and when λ_c is minimized as in (15) Δ has double zeros at $\omega = \pm 1$.]

Solving (43) by Cramer's rule we obtain

$$\tilde{g}_1(\omega) = \frac{1}{\Delta}\{\alpha_1(\omega)[(\omega^2\zeta+1)^2-\lambda_c(\omega^2\zeta+\tfrac{1}{2}\alpha)]-\beta_1(\omega)(1+\zeta)^2\} \tag{46}$$

$$\tilde{h}_1(\omega) = \frac{1}{\Delta}\{\beta_1(\omega)(\omega^2\zeta+1)^2-\alpha_1(\omega)[(\zeta+1)^2-\lambda_c(\zeta+\tfrac{1}{2}\alpha)]\} \tag{47}$$

We note that $\tilde{g}(\omega)$ and $\tilde{h}(\omega)$ are analytic in the neighborhood of $\omega = \pm 1$ because of the requirement (20) concerning the exponential decay of W_0. Consequently for the absence of the double pole at $\omega = \pm 1$ we require that

$$\alpha_1(1)[(\zeta+1)^2-\lambda_c(\zeta+\tfrac{1}{2}\alpha)]-\beta_1(1)(1+\zeta)^2 = 0, \tag{48}$$

and

$$\alpha_1'(1)[(\zeta+1)^2-\lambda_c(\zeta+\tfrac{1}{2}\alpha)]-\beta_1'(1)(\zeta+1)^2-\beta_1(1)4(\zeta+1)\zeta = 0, \tag{49a}$$

$$\alpha_1'(1)[(\zeta+1)^2-\lambda_c(\zeta+\tfrac{1}{2}\alpha)]+\alpha_1(1)([4\zeta(\zeta+1)-\lambda_c 2\zeta]-\beta_1'(1)(\zeta+1)^2 = 0, \tag{49b}$$

where

$$\alpha_1'(1) = \left.\frac{d\alpha_1}{d\omega}\right|_{\omega=1} \quad \text{and} \quad \beta_1'(1) = \left.\frac{d\beta_1}{d\omega}\right|_{\omega=1}$$

Equations (48), (49a) and (49b) are not independent, for, if we subtract (49a) from (49b) and use relation (15) for λ_c we retrieve (48), thus we have only two independent conditions for the suppression of the double poles. These two conditions, taken together with the continuity requirements (33), and equation (44) are sufficient to determine B_1 and the jump $[a_1(0)]$ in $a_1(z)$.

To proceed, we apply continuity relations (33), with $m = 1$, to the solution (39) at $y = 0$ and get

$$\begin{aligned} [g_1(0)] &= -[a_1(0)]-[\bar{a}_1(0)] = -[h_1(0)], \\ [g_1'(0)] &= -i[a_1(0)]+i[\bar{a}_1(0)] = -[h_1'(0)], \\ [g_1''(0)] &= [a_1(0)]+[\bar{a}_1(0)] = -[h_1''(0)], \\ [g_1'''(0)] &= i[a_1(0)]-i[\bar{a}_1(0)] = -[h_1'''(0)]. \end{aligned} \tag{50}$$

Substituting these relations into (44) and setting $\omega = 1$ we find

$$\begin{aligned} \alpha_1(1) &= -\zeta(\zeta+1)4i[\bar{a}_1(0)], \\ \beta_1(1) &= \lambda_c\{(\zeta+\tfrac{1}{2}\alpha)B_1\tilde{W}_0(1)-2\zeta i[\bar{a}_1(0)]\}-\alpha(1), \end{aligned} \tag{51}$$

and substituting these into (48) gives

$$(\zeta+\tfrac{1}{2}\alpha)(1+\zeta)B_1\tilde{W}_0(1) = 0.$$

Assuming $\tilde{W}_0(1) \neq 0$ [the case $\tilde{W}_0(1) = 0$ is discussed briefly in the concluding remarks] and noting that $\zeta > 0$ we must have

$$B_1 = 0. \tag{52}$$

Similarly (49a) [or (49b)] can be used to show that

$$\{3\lambda_c\zeta-8\zeta(\zeta+1)\}[\bar{a}_1(0)] = 0$$

which implies that

$$[a_1(0)] = 0 \tag{53}$$

and hence from (50) and (44)

$$\alpha_1(\omega) \equiv 0, \qquad \beta_1(\omega) \equiv 0, \tag{54}$$

and from (46), (47) and (54)

$$\tilde{g}_1(\omega) \equiv 0, \qquad \tilde{h}_1(\omega) \equiv 0. \tag{55}$$

Therefore

$$g_1(y) \equiv 0, \quad \text{and } h_1(y) \equiv 0, \tag{56}$$

so that finally

$$\begin{aligned} f_1 &= [a_1(z)\,\mathrm{e}^{iy} + \bar{a}_1(z)\,\mathrm{e}^{-iy}]\sin x, \\ w_1 &= -[a_1(z)\,\mathrm{e}^{iy} + \bar{a}_1(z)\,\mathrm{e}^{-iy}]\sin x. \end{aligned} \tag{57}$$

5. SOLUTION OF SECOND ORDER PERTURBATION EQUATION

Substitution of results (52) and (57) into the right hand sides of (29) gives

$$\begin{aligned} L^{(1)}(f_2, w_2) &= -(1+\zeta)^2 H[a_1^2\,\mathrm{e}^{2iy} + \bar{a}_1^2\,\mathrm{e}^{-2iy} - 2a_1\bar{a}_1\cos 2x], \\ L^{(2)}(f_2, w_2) &= \lambda_c[\tfrac{1}{2}\alpha W_0(y) - \zeta W_0''(y)]B_2\sin x \\ &\quad + 2K(\zeta)H[a_1^2\,\mathrm{e}^{2iy} + \bar{a}_1^2\,\mathrm{e}^{-2iy} - 2a_1\bar{a}_1\cos 2x], \end{aligned} \tag{58}$$

and the solution of (58) contains no secular terms because the terms that are linear in f_1 and w_1 on the right of (29) satisfy condition (38) identically. The solution of (58) is clearly similar to the first order solution except for the addition of a further particular solution, due to the terms proportional to $[a_1^2\,\mathrm{e}^{2iy} + \bar{a}_1^2\,\mathrm{e}^{-2iy} - 2a_1\bar{a}_1\cos 2x]$ on the right. This particular solution is similar to that obtained by Budiansky and Amazigo in Ref. [5] for their second order equations. Thus we write, for the solution of (58),

$$\begin{aligned} f_2 &= [a_2(z)\,\mathrm{e}^{iy} + \bar{a}_2(z)\,\mathrm{e}^{-iy}]\sin x + g_2(y)\sin x \\ &\quad + f_{20}(x) + f_{22}(x)[a_1^2\,\mathrm{e}^{2iy} + \bar{a}_1^2\,\mathrm{e}^{-2iy}], \end{aligned} \tag{59}$$

$$\begin{aligned} w_2 &= -[a_2(z)\,\mathrm{e}^{iy} + \bar{a}_2(z)\,\mathrm{e}^{-iy}]\sin x + h_2(y)\sin x \\ &\quad + w_{20}(x) + w_{22}(x)[a_1^2\,\mathrm{e}^{2iy} + \bar{a}_1\,\mathrm{e}^{-2iy}], \end{aligned} \tag{60}$$

where the first term in (59) and (60) is the solution to the homogeneous equations, $g_2(y)$, $h_2(y)$ is a particular solution obtained in the same way as $g_1(y)$ and $h_1(y)$, and the remaining terms give the form of the new particular solution.

First, however, we note that $g_2(y)$ and $h_2(y)$ must satisfy equations (40)–(49b) of the last section, with subscript 1 replaced everywhere by subscript 2. The continuity condition at $y = 0, z = 0$ will be different, and we notice that since $a_1(z)$ is continuous, the new particular solution does not enter into it, so applying (33) with $m = 2$ we obtain

$$\begin{aligned} [g_2(0)] &= -[a_2(0)] - [\bar{a}_2(0)] = -[h_2(0)], \\ [g_2'(0)] &= -i[a_2(0)] + i[\bar{a}_2(0)] - [a_1'(0)] - [\bar{a}_1'(0)] = -[h_2'(0)], \\ [g_2''(0)] &= [a_2(0)] + [\bar{a}_2(0)] - 2i[a_1'(0)] + 2i[\bar{a}_1'(0)] = -[h_2''(0)], \\ [g_2'''(0)] &= i[a_2(0)] - i[\bar{a}_2(0)] + 3[a_1'(0)] + 3[\bar{a}_1'(0)] = -[h_2'''(0)]. \end{aligned} \tag{61}$$

Hence, using (44) gives

$$\begin{aligned} \alpha_2(1) &= -\zeta(\zeta+1)4i[\bar{a}_2(0)] + 2\zeta^2\{[a_1'(0)] + 3[\bar{a}_1'(0)]\} + 2\zeta\{[a_1'(0)] + [\bar{a}_1'(0)]\} \\ \beta_2(1) &= \lambda_c\{(\zeta + \tfrac{1}{2}\alpha)B_2\tilde{W}_0(1) - 2\zeta i[\bar{a}_2(0)] + \zeta([a_1'(0)] + [\bar{a}_1'(0)])\} - \alpha_2(1) \end{aligned} \tag{62}$$

and condition (48) for the absence of the double pole gives

$$(\zeta+1)(\zeta+\tfrac{1}{2}\alpha)B_2\tilde{W}_0(1) = 2\zeta^2[\bar{a}_1'(0)]. \tag{63}$$

(Note, that since $W_0(y)$ is a real function $\bar{\tilde{W}}_0(1) = \tilde{W}_0(-1)$ and $(\zeta+1)(\zeta+\tfrac{1}{2}\alpha)B_2\tilde{W}_0(-1) = 2\zeta^2[a_1'(0)]$.) The other condition for the absence of the double pole, either (49a) or (49b), can be used to obtain an expression for $[a_2(0)]$ in terms of B_2 and $\tilde{W}_0(1)$, however, this expression will not be required subsequently so we omit it. We have not of course determined $g_2(y)$ and $h_2(y)$, but since these terms do not contribute to the secular part of the solution to the third perturbation equation (30) they are not required.

We find the other part of the particular solution by substituting (59) and (60) into (58) to obtain the ordinary differential equations for $f_{20}, w_{20}, f_{22}, w_{22}$, which are

$$\begin{aligned} f_{20}'''' - (1+\zeta)^2 w_{20}'' &= 2(1+\zeta)^2 H a_1 \bar{a}_1 \cos 2x \\ w_{20}'''' + \tfrac{1}{2}\alpha\lambda_c w_{20}'' - K(\zeta) f_{20}'' &= -4HK(\zeta) a_1 \bar{a}_1 \cos 2x \end{aligned} \tag{64}$$

and

$$\begin{aligned} f_{22}'''' - 8\zeta f_{22}'' + 16\zeta^2 f_{22} - (1+\zeta)^2 w_{22}'' &= -(1+\zeta)^2 H \\ w_{22}'''' + (\tfrac{1}{2}\alpha\lambda_c - 8\zeta) w_{22}'' + (16\zeta^2 - 4\zeta\lambda_c) w_{22} - K(\zeta) f_{22}'' &= 2HK(\zeta) \end{aligned} \tag{65}$$

where $(\ )' = \dfrac{d}{dx}(\ )$.

Except for the normalization and the z dependent factor $a_1(z)$ equations (64) and (65) are the same as equations (38) and (39) of Ref. [5] and are solved in the same way. The solution of (64) is

$$\begin{aligned} w_{20} &= \sum_{m=1,3,5\ldots}^{\infty} a_1 \bar{a}_1 d_m \sin mx \\ f_{20} &= (1+\zeta)^2 \left\{ \frac{1}{8} H a_1 \bar{a}_1 (2x^2 + \cos 2x - 1) - \sum_{m=1,3,5\ldots}^{\infty} \frac{a_1 \bar{a}_1 d_m}{m^2} \sin mx \right\} \end{aligned} \tag{66}$$

where

$$d_m = (8A^2 H/\pi m)(m^2-4)^{-1}(m^4 - \tfrac{1}{2}\alpha\lambda_c m^2 + A^2)^{-1}[1 + 2m^2(1+\zeta)^{-2}] \tag{67}$$

and the solution of (65) is

$$w_{22} = \sum_{m=1,3,5\ldots}^{\infty} b_m \sin mx, \qquad f_{22} = \sum_{m=1,3,5\ldots}^{\infty} c_m \sin mx \tag{68}$$

where

$$b_m = -\frac{4HA^2[2(m^2+4\zeta)^2(1+\zeta)^{-2} + m^2]}{\pi m\{(m^2+4\zeta)^2[(m^2+4\zeta)^2 - \tfrac{1}{2}\alpha\lambda_c m^2 - 4\zeta\lambda_c] + A^2 m^4\}}, \tag{69}$$

and

$$c_m = \frac{4H\{2A^2m^2 - (1+\zeta)^2[(m^2+4\zeta)^2 - \tfrac{1}{2}\alpha\lambda_c m^2 - 4\zeta\lambda_c]\}}{\pi m\{(m^2+4\zeta)^2[(m^2+4\zeta)^2 - \tfrac{1}{2}\alpha\lambda_c m^2 - 4\zeta\lambda_c] + A^2 m^4\}}. \tag{70}$$

To summarize, the solution of the second order perturbation equation is

$$f_2 = [a_2(z)\,e^{iy}+\bar{a}_2(z)\,e^{-iy}]\sin x+(1+\zeta)^2\tfrac{1}{8}Ha_1\bar{a}_1(2x^2+\cos 2x-1)$$
$$+\sum_{m=1,3,5\ldots}^{\infty}\left\{-(1+\zeta)^2\frac{d_m a_1\bar{a}_1}{m^2}+[a_1^2\,e^{2iy}+\bar{a}_1^2\,e^{-2iy}]c_m\right\}\sin mx+g_2(y)\sin x, \tag{71}$$

$$w_2 = -[a_2(z)\,e^{iy}+\bar{a}_2(z)\,e^{-iy}]\sin x$$
$$+\sum_{m=1,3,5\ldots}^{\infty}\{a_1\bar{a}_1 d_m+[a_1^2\,e^{2iy}+\bar{a}_1^2\,e^{-2iy}]b_m\}\sin mx+h_2(y)\sin x. \tag{72}$$

6. ELIMINATION OF SECULAR TERMS IN THIRD ORDER PERTURBATION EQUATION

Finally, eliminating the secular terms from the solution to the third order perturbation equations (30) provides a differential equation for $a_1(z)$, from which an approximate load–deflection equation can be obtained.

Let R_1 and R_2 be the nonhomogeneous terms in the first and second respectively of equations (30) that contribute secular terms in the solution. Recall that $B_1 = 0$ [equation (52)], and notice that the terms linear in f_2, w_2 that would contribute secular terms satisfy (38) identically, and of course terms involving $W_0(y)$, $h_2(y)$ or $g_2(y)$ do not contribute. Also, terms proportional to $w_1 w_1$ and $w_1 f_1$ (such as $w_{1,yz}w_{1,xx}$) do not contribute, so that finally we need only examine the expressions

$$-2\zeta\frac{\partial^2}{\partial z^2}\left(\nabla^2+2\zeta\frac{\partial^2}{\partial y^2}\right)f_1-(1+\zeta)^2HS(w_1, w_2) \tag{73}$$

and

$$-2\zeta\frac{\partial^2}{\partial z^2}\left(\nabla^2+2\zeta\frac{\partial^2}{\partial y^2}+\frac{1}{2}\lambda_c\right)w_1+\lambda_c(\tfrac{1}{2}\alpha w_{1,xx}+\zeta w_{1,yy})-HK(\zeta)\{S(w_1, f_2)+S(f_1, w_2)\} \tag{74}$$

in calculating R_1 and R_2 respectively. For example, in the expression $S(w_1, w_2) = -S(f_1, w_2)$ (because $w_1 = -f_1$) the term $w_{1,xy}w_{2,xy}$ appears and we have that

$$\begin{aligned} w_{1,xy}w_{2,xy} &= -i(a_1\,e^{iy}-\bar{a}_1\,e^{-iy})\cos x\sum_{m=1,3,5\ldots}^{\infty}2imb_m[a_1^2\,e^{2iy}-\bar{a}_1^2\,e^{-2iy}]\cos mx \\ &\quad+[\text{non-secular terms (n.s.t.)}] \\ &= -(a_1^2\bar{a}_1\,e^{iy}+\bar{a}_1^2 a_1\,e^{-iy})\sum_{m=1,3,5\ldots}^{\infty}mb_m[\cos(m+1)x+\cos(m-1)x] \\ &\quad+(\text{n.s.t.}) \end{aligned} \tag{75}$$

and noting that for even integers p

$$\cos px = -\frac{4}{\pi(p^2-1)}\sin x-\sum_{n=3,5\ldots}^{\infty}\frac{4n\sin mx}{\pi(p^2-n^2)}, \tag{76}$$

we obtain

$$w_{1,xy}w_{2,xy} = (a_1^2\bar{a}_1\,e^{iy}+\bar{a}_1^2a_1\,e^{-iy})\sum_{m=1,3,5\ldots}^{\infty}\frac{8mb_m}{\pi(m^2-4)}\sin x+(\text{n.s.t.}). \tag{77}$$

Thus we find that

$$R_1 = 2\zeta(1+3\zeta)(a_1''\,e^{iy}+\bar{a}_1''\,e^{-iy})\sin x$$
$$+(1+\zeta)^2H(a_1^2\bar{a}_1\,e^{iy}+\bar{a}_1^2a_1\,e^{-iy})\sum_{m=1,3,5\ldots}^{\infty}\frac{8}{\pi m(m^2-4)}\{(m^2-4)b_m-m^2d_m\}\sin x, \tag{78}$$

and

$$R_2 = -2\zeta(1+3\zeta-\tfrac{1}{2}\lambda_c)(a_1''\,e^{iy}+\bar{a}_1''\,e^{-iy})\sin x+(\zeta+\tfrac{1}{2}\alpha)\lambda_c(a_1\,e^{iy}+\bar{a}_1\,e^{-iy})\sin x$$
$$-K(\zeta)H(a_1^2\bar{a}_1\,e^{iy}+\bar{a}_1^2a_1\,e^{-iy})\left\{(1+\zeta)^2\frac{3}{4}H+\sum_{m=1,3,5\ldots}^{\infty}\frac{8}{\pi m}[b_m-c_m]\right.$$
$$\left.-\sum_{m=1,3,5\ldots}^{\infty}\frac{8}{\pi m(m^2-4)}[m^2+(1+\zeta)^2]d_m\right\}\sin x, \tag{79}$$

where $(\ )' = \dfrac{d}{dz}(\ )$.

Now substituting the parts of (78) and (79) proportional to $e^{+iy}\sin x$ into (38) we obtain

$$(1+\zeta)^{-1}2\zeta^2a_1''-(\zeta+\tfrac{1}{2}\alpha)a_1-4(\zeta+\tfrac{1}{2}\alpha)ba_1^2\bar{a}_1 = 0, \tag{80}$$

and from the parts proportional to $e^{-iy}\sin x$ we obtain the complex conjugate of (80), where

$$\frac{b}{1-\nu^2} = \frac{24\zeta^2}{\lambda_c(\zeta+\frac{1}{2}\alpha)}\left\{\frac{3}{32}-\frac{8A^2}{\pi^2}\sum_{m=1,3,5\ldots}^{\infty}\frac{[1+2m^2(1+\zeta)^{-2}]^2}{m^2(m^2-4)^2(m^4-\frac{1}{2}\alpha\lambda_cm^2+A^2)}\right.$$
$$\left.-\frac{4}{\pi^2}\sum_{m=1,3,5\ldots}^{\infty}\frac{(m^2+4\zeta)^2[4A^2(1+\zeta)^{-4}-1]+4A^2m^2(1+\zeta)^{-2}+\frac{1}{2}\alpha\lambda_cm^2+4\zeta\lambda_c}{m^2\{(m^2+4\zeta)^2[(m^2+4\zeta)^2-\frac{1}{2}\alpha\lambda_cm^2-4\zeta\lambda_c]+A^2m^4\}}\right\} \tag{81}$$

is identical to the same quantity defined by equation (58)† and plotted in Fig. 3 of Ref. [5].

Hence, $a_1(z)$ must satisfy the differential equation (80) for $|z| > 0$ together with the jump relation (63) at $z = 0$, and a boundedness condition at $z = \pm\infty$. Equations (80) and (63) have the same form as the corresponding equations (Equations (53) and (52), respectively, of Ref. [4]) for the beam on a nonlinear elastic foundation with a dimple shaped initial deflection, and will therefore be analyzed in the same manner. Thus multiplying (80) by $\bar{a}_1'(z)$ and adding the result to its conjugate gives

$$(1+\zeta)^{-1}2\zeta^2(a_1'\bar{a}_1')'-(\zeta+\tfrac{1}{2}\alpha)(a_1\bar{a}_1)'-2(\zeta+\tfrac{1}{2}\alpha)b(a_1^2\bar{a}_1^2)' = 0. \tag{82}$$

Integrating from 0^+ to ∞, assuming $a_1(\infty) = a_1'(\infty) = 0$, gives

$$(1+\zeta)^{-1}2\zeta^2a_1'(0^+)\bar{a}_1'(0^+)-(\zeta+\tfrac{1}{2}\alpha)|a_1(0)|^2-2(\zeta+\tfrac{1}{2}\alpha)b|a_1(0)|^4 = 0. \tag{83}$$

† Equation (58) of Ref. [5] contains misprints.

But note that if $a_1(z)$ satisfies (80) in $(0^+, \infty)$, $a_1(\infty) = 0$ and $a_1(z)$ is continuous at $z = 0$, then $a_1(-z)$ is the solution in $(-\infty, 0^-)$. Hence

$$a_1'(0^+) = -a_1'(0^-) = \tfrac{1}{4}\zeta^{-2}(\zeta+1)(\zeta+\tfrac{1}{2}\alpha)B_2\tilde{W}_0(-1) \tag{84}$$

with the use of (63). Thus (83) becomes

$$\tfrac{1}{8}\zeta^{-2}(\zeta+1)(\zeta+\tfrac{1}{2}\alpha)B_2^2|\tilde{W}_0(1)|^2 = |a_1(0)|^2+2b|a_1(0)|^4. \tag{85}$$

We can now write down the first terms in the expansions (24) and (25) which are

$$f = \eta[a_1(z)\,e^{iy}+\bar{a}_1(z)\,e^{-iy}]\sin x, \tag{86}$$

$$w = -\eta[a_1(z)\,e^{iy}+\bar{a}_1(z)\,e^{-iy}]\sin x, \tag{87}$$

$$\lambda\varepsilon = \eta^2\lambda_c B_2, \tag{88}$$

and introducing a displacement amplitude parameter $\sigma = \eta|a_1(0)|$ we have from (85) that

$$B_2^2 = 8\zeta^2(1+\zeta)^{-1}(\zeta+\tfrac{1}{2}\alpha)^{-1}|\tilde{W}_0(1)|^{-2}\left\{\frac{\sigma^2}{\eta^2}+2b\frac{\sigma^2}{\eta^2}\right\}, \tag{89}$$

and using (88)

$$\begin{aligned}(\lambda/\lambda_c)^2\varepsilon^2|\tilde{W}_0(1)|^2 &= 8\zeta^2(1+\zeta)^{-1}(\zeta+\tfrac{1}{2}\alpha)^{-1}(\eta^2\sigma^2+2b\sigma^4)\\ &= 8\zeta^2(1+\zeta)^{-1}(\zeta+\tfrac{1}{2}\alpha)^{-1}[(1-\lambda/\lambda_c)\sigma^2+2b\sigma^4]\end{aligned} \tag{90}$$

using (21). Equation (90) is an approximate relation between load parameter λ and displacement parameter σ, and if $b > 0$, λ will continue to increase after λ reaches λ_c but if $b < 0$, λ will reach a maximum $\bar{\lambda} < \lambda_c$ given by

$$\frac{\bar{\lambda}}{\lambda_c} = \{1+\zeta^{-1}[(\zeta+1)(\zeta+\tfrac{1}{2}\alpha)(-b)]^{\frac{1}{2}}\varepsilon|\tilde{W}_0(1)|\}^{-1} \tag{91}$$

$$\approx 1-\zeta^{-1}[(\zeta+1)(\zeta+\tfrac{1}{2}\alpha)(-b)]^{\frac{1}{2}}\varepsilon|\tilde{W}_0(1)|. \tag{92}$$

As in Ref. [4] we expect that expression (91) for $\bar{\lambda}/\lambda_c$ is more accurate than expression (92). Thus the shell is imperfection-sensitive to the dimple shaped initial deflection [equation (19)] whenever b is negative, which is also the case for modal imperfections [5].

7. CONCLUDING REMARKS

For purposes of comparison we exhibit the asymptotic relations between $\bar{\lambda}$ and ε for the two types of imperfections, namely localized dimple and modal imperfections.

(i) Dimple imperfection: $w_0(x, y) = \varepsilon W_0(y)\sin x$ with

$|W_0(y)| < M\,e^{-\alpha|y|}, M, \alpha > 0$:

$$1-\bar{\lambda}/\lambda_c \approx B|\tilde{W}_0(1)|\varepsilon\bar{\lambda}/\lambda_c \tag{93}$$

where

$$B = \frac{1}{\zeta}\left[(-b)(\zeta+1)\left(\zeta+\frac{\alpha}{2}\right)\right]^{\frac{1}{2}}$$

$$\lambda_c = \frac{4(1+\zeta)^2}{3\zeta+1+\alpha}, \qquad Z^2 = \frac{\pi^4(1+\zeta)^4(\zeta-1+\alpha)}{12(3\zeta+1+\alpha)}$$

and

$$\tilde{W}_0(1) = \int_{-\infty}^{\infty} W_0(y)\,e^{iy}\,dy$$

(ii) Modal imperfection: $w_0(x, y) = \varepsilon \sin y \sin x$

$$(1-\bar{\lambda}/\lambda_c)^{\frac{3}{2}} \approx \frac{3\sqrt{3}}{2}\sqrt{(-b)}\varepsilon\bar{\lambda}/\lambda_c \tag{94}$$

or

$$1-\bar{\lambda}/\lambda_c \approx 3(2)^{-\frac{2}{3}}(-b)^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}(\bar{\lambda}/\lambda_c)^{\frac{2}{3}}.$$

The dependence of B and $(-b)^{\frac{1}{3}}$ on Z is plotted in Fig. 1 for lateral pressure ($\alpha = 0$) and hydrostatic pressure ($\alpha = 1$) for b negative. It is to be noted that B remains finite as $Z \to 0$. The degrees of imperfection-sensitivity are governed by B and $(-b)^{\frac{1}{3}}$ in a manner given by the plots in Fig. 2. Here $\bar{\lambda}/\lambda_c$ is plotted against $\varepsilon|\tilde{W}_0(1)|$, and against ε for dimple and modal imperfections respectively.

The result (93) holds for all dimple imperfections for which $\tilde{W}_0(1) \neq 0$. However it breaks down if $\tilde{W}_0(1) = 0$ and it is of limited value if $\tilde{W}_0(1)$ is very small compared to $\tilde{W}_0(\alpha)$

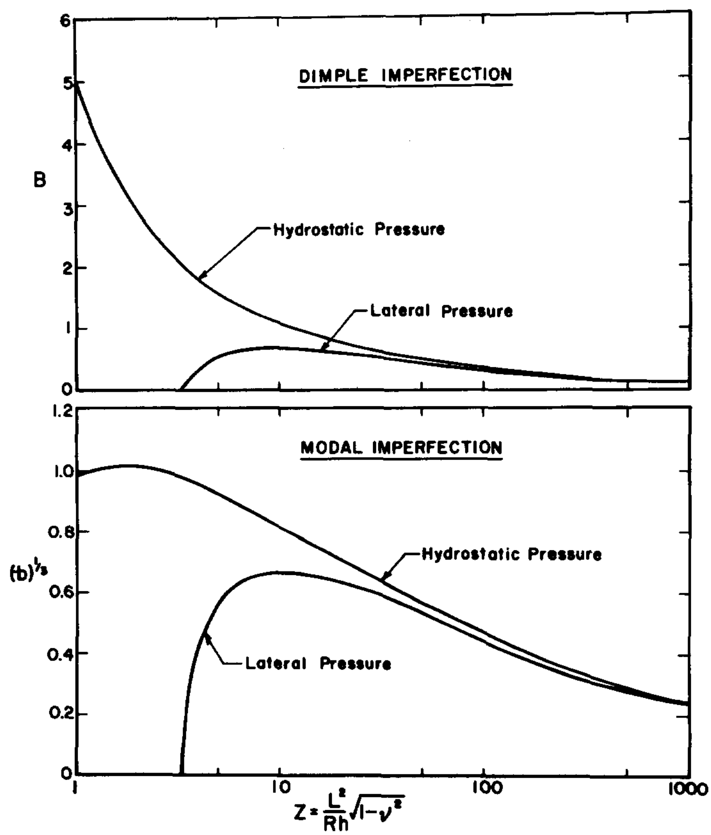

FIG. 1.

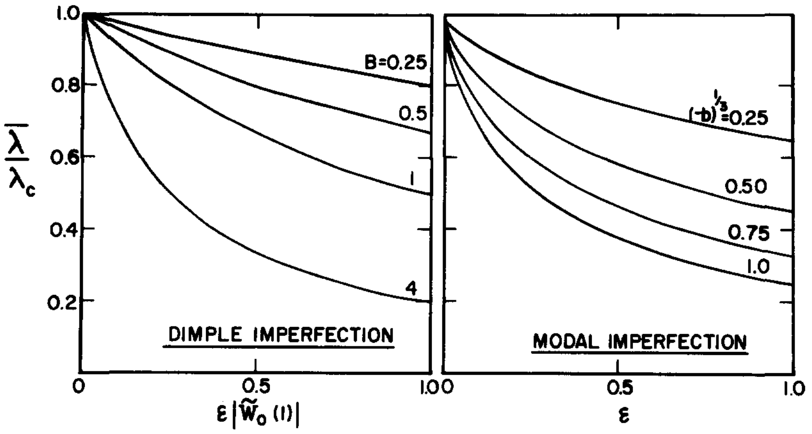

FIG. 2.

for values of α near unity. The breakdown of (93) occurs because in such cases buckling is triggered by other components of W_0 besides $\tilde{W}_0(1)$.

Finally from equations (93) and (94) we note that the degradation in the buckling load due to dimple shaped imperfection is directly proportional to the imperfection amplitude ε, whereas for the modal imperfection the degradation is proportional to $\varepsilon^{\frac{2}{3}}$.

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APPENDIX

Here for completeness we show briefly how the results of Ref. [5] for a modal initial imperfection may be obtained using perturbation expansion in the load parameter η. Let us take

$$w_0(x, y) = \varepsilon \sin y \sin x \tag{95}$$

[i.e. set $W_0(y) = \sin y$ in equation (19) and relax condition (20)]. Again we define a load parameter η as in (21), but in this case it is not necessary to introduce the stretched variable z. Thus we expand in powers of η as in (24)–(26) but drop the z dependence from these expansions so that the first three perturbation equations are the same as (28)–(30) with the derivatives of quantities with respect to z set equal to zero. This leads to considerable simplification of the right hand sides of (29) and (30). With $W_0(y) = \sin y$, equations (28) become

$$\begin{aligned} L^{(1)}(f_1, w_1) &= 0 \\ L^{(2)}(f_1, w_1) &= \lambda_c B_1(\zeta + \tfrac{1}{2}\alpha) \sin y \sin x \end{aligned} \tag{96}$$

and the suppression of secular terms gives

$$B_1 = 0 \tag{97}$$

and the solution of (96) is therefore

$$\begin{aligned} f_1 &= a \sin y \sin x \\ w_1 &= -a \sin y \sin x \end{aligned} \tag{98}$$

where a is a constant.

Substituting (97) and (98) into the right hand side of (29) we obtain

$$\begin{aligned} L^{(1)}(f_2, w_2) &= (1+\zeta)^2 H \tfrac{1}{2} a^2 (\cos 2x + \cos 2y) \\ L^{(2)}(f_2, w_2) &= \lambda_c B_2(\zeta + \tfrac{1}{2}\alpha) \sin y \sin x - HK(\zeta) a^2 (\cos 2x + \cos 2y) \end{aligned} \tag{99}$$

and suppression of secular terms gives

$$B_2 = 0.$$

If we also note that (99) can be obtained from (58), by setting

$$a_1 = \frac{1}{2i} a = -\bar{a}_1 \tag{100}$$

so that the particular integral of (99) is given by equations (64)–(70) with the substitution (100), then finally

$$\begin{aligned} f_2 &= (1+\zeta)^2 \tfrac{1}{4} a^2 \left\{ \frac{1}{8} H(2x^2 + \cos 2x - 1) - \sum_{m=1,3,5\ldots}^{\infty} \frac{d_m}{m^2} \sin mx \right\} \\ &\quad - \left(\sum_{m=1,3,5\ldots}^{\infty} c_m \sin mx \right) \tfrac{1}{2} a^2 \cos 2y \\ w_2 &= \sum_{m=1,3,5\ldots}^{\infty} \tfrac{1}{4} a^2 d_m \sin mx - \left(\sum_{m=1,3,5\ldots}^{\infty} b_m \sin mx \right) \tfrac{1}{2} a^2 \cos 2y. \end{aligned} \tag{101}$$

Finally B_3 is determined by eliminating the secular terms from the third perturbation equations. Let R_1 and R_2 be the secular terms on the right of the first and second of equations (30) respectively, then

$$R_1 = (1+\zeta)^2 H \sum_{m=1,3,5\ldots}^{\infty} \frac{2a^3}{\pi m(m^2-4)} \{(m^2-4) b_m - m^2 d_m\} \sin y \sin x \tag{102}$$

$$\begin{aligned} R_2 &= \lambda_c(\zeta + \tfrac{1}{2}\alpha)(B_3 + a) \sin y \sin x - HK(\zeta)(1+\zeta)^2 \tfrac{3}{16} H a^3 \sin y \sin x \\ &\quad + HK(\zeta) \sum_{m=1,3,5\ldots}^{\infty} \frac{2a^3}{\pi m(m^2-4)} \{[1+\zeta)^2 + m^2] d_m (m^2-4)(b_m - c_m)\} \sin y \sin x. \end{aligned} \tag{103}$$

Substituting from these two expressions into condition (38) for the suppression of secular terms and using (3), (67), (69), (70) and (81) we get

$$B_3 = -a-ba^3. \tag{104}$$

Thus for the modal imperfection (95) the first terms in the expansions (24) and (25) are

$$\begin{aligned} f &= \eta a \sin x \sin y \\ w &= -\eta a \sin x \sin y \end{aligned} \tag{105}$$

$$(\lambda/\lambda_c)\varepsilon = -a\eta^3 - ba^3\eta^3 \tag{106}$$

and introducing a displacement amplitude parameter $\sigma = -a\eta$ and using (21) gives

$$(\lambda/\lambda_c)\varepsilon = \sigma(1-\lambda/\lambda_c)+b\sigma^3 \tag{107}$$

or

$$\lambda/\lambda_c = \frac{\sigma}{\sigma+\varepsilon}\{1+b\sigma^2\} \tag{108}$$

and from equation (107) we see that if $b < 0$, λ will have a maximum value $\bar{\lambda} < \lambda_c$ given by

$$(1-\bar{\lambda}/\lambda_c)^{\frac{3}{2}} = \frac{3\sqrt{3}}{2}\varepsilon\sqrt{(-b)}\bar{\lambda}/\lambda_c \tag{109}$$

in agreement with the results of Ref. [5].

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Абстракт—Определяется приближенное поведение нагрузка-прогиб свободно опертой, цилиндрической оболочки, подверженной действию внешего давления, для случая, в котором оболочк имеет начальный прогиб в форме вмятины. Это достигается путем использования "двухсинхронного" разложения возмущения, примененного к уравнениям Кармана-Донелла для оболочек. Показано, что если оболочка чувствитеьна к неправильностям для начальных прогибов, характеризованных формой волнообразования при линейном выпучивании, тогда она также чувствительна к неправильности типа вмятины. Уменьшение нагрузки выпучивания для неправильности типа вмятины зависит линейно от амплитуды начального прогиба ϵ, тогда как для случая модальной неправильности оно пропорциально к степени 2/3 ϵ.